Entanglement classification of restricted Greenberger-Horne-Zeilinger–symmetric states in a four-qubit system

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(Received 9 May 2013; published 27 May 2014)

Similar to the three-qubit Greenberger-Horne-Zeilinger (GHZ) symmetry we explore the four-qubit GHZ symmetry group and its subgroup, the restricted GHZ symmetry group. While the set of symmetric states under the whole group transformation is represented by three real parameters, the set of symmetric states under the subgroup transformation is represented by two real parameters. After comparing the symmetric states for whole group and subgroup, the entanglement is examined for the latter set. It is shown that the set has only two stochastic local operations and classical communication classes, L_{abc_2} and G_{abcd}. Extension to the multiqubit system is briefly discussed.

DOI: 10.1103/PhysRevA.89.052326 PACS number(s): 03.67.Mn, 03.65.Ud

I. INTRODUCTION

Quantum entanglement [1] is the most important notion in quantum technology (QT) and quantum information theory (QIT). As shown for the past two decades it plays a crucial role in quantum teleportation [2], superdense coding [3], quantum cloning [4], and quantum cryptography [5]. It is also quantum entanglement that makes the quantum computer outperform the classical one [6,7]. Thus, in order to develop QT and QIT it is essential to understand how to quantify and how to characterize the multipartite entanglement.

Since quantum entanglement is a nonlocal property of a given multipartite quantum state, it should be invariant under the local unitary (LU) transformations, i.e., the unitary operations acted independently on each of the subsystems. If $|\psi\rangle$ and $|\varphi\rangle$ are in the same category in the LU, one state can be obtained with certainty from the other one by means of local operations and classical communication (LOCC) [8,9]. This implies that $|\psi\rangle$ and $|\varphi\rangle$ can be used, respectively, to implement the same task of QIT with equal probability of successful performance of the task. However, the classification of entanglement through LU generates infinite equivalence classes even in the simplest bipartite systems.

In order to escape this difficulty the classification through stochastic local operations and classical communication (SLOCC) was suggested in Ref. [8]. If $|\psi\rangle$ and $|\varphi\rangle$ are in the same SLOCC class, one state can be converted into the other state with nonzero probability by means of LOCC. This fact implies that $|\psi\rangle$ and $|\varphi\rangle$ can be used, respectively, to implement the same task of QIT although the probability of success for this task is different. Mathematically, if two n-party states $|\psi\rangle$ and $|\varphi\rangle$ are in the same SLOCC class, they are related to each other by $|\psi\rangle = A_1 \otimes A_2 \otimes \cdots \otimes A_n |\varphi\rangle$ with $\{A_j\}$ being arbitrary invertible local operators.[1] However, it is more useful to restrict ourselves to the SLOCC transformation where all $\{A_j\}$ belong to SL(2, C), the group of 2×2 complex matrices having determinant equal to 1.

The SLOCC classification was first examined in the three-qubit pure-state system [10]. It was shown that the whole system consists of six inequivalent SLOCC classes, i.e., fully separable (S), three biseparable (B), W, and Greenberger-Horne-Zeilinger (GHZ) classes. Moreover, it is possible to know which class an arbitrary state $|\psi\rangle$ belongs in by computing the residual entanglement [11] and concurrences [12] for its partially reduced states. Similarly, the entanglement of whole three-qubit mixed states consists of S, B, W, and GHZ types [13]. It was shown that these classes satisfy a linear hierarchy $S \subset B \subset W \subset \text{GHZ}$.

Generally, a given QIT task requires a particular type of entanglement. In addition, the effect of the environment generally converts the pure state prepared for the QIT task into the mixed state. Therefore, it is important to distinguish the entanglement of mixtures to perform the QIT task successfully. However, it is a notoriously difficult problem to know which type of entanglement is contained in the given multipartite mixed state. Even for a three-qubit state it is a very difficult problem because analytical computation of the residual entanglement for arbitrary mixed states has been generally impossible so far.[2]

Recently, classification of the entanglement classes for three-qubit mixed states has significantly progressed. In Ref. [15] the GHZ symmetry was examined in a three-qubit system. This is a symmetry that GHZ states $|\text{GHZ}_3\rangle_\pm = (1/\sqrt{2})(|000\rangle \pm |111\rangle)$ have up to the global phase and is expressed as a symmetry under (i) qubit permutations, (ii) simultaneous flips, and (iii) qubit rotations about the z axis. All the GHZ-symmetric states can be parametrized by two real parameters, say, x and y. Authors in Ref. [15] succeeded in classifying the entanglement of the GHZ-symmetric states completely. This complete classification makes it possible to compute the three-tangle[3] analytically for all the GHZ-symmetric states [16] and to construct the class-specific optimal witnesses [17]. It also makes it possible to obtain

[1]For a complete proof on the connection between SLOCC and local operations see Appendix A of Ref. [10].

[2]However, it is possible to compute the residual entanglement for a few rare cases [14].

[3]The definition of three-tangle in this paper is the square root of the residual entanglement presented in Ref. [11].

the lower bound of the three-tangle for an arbitrary three-qubit mixed state [18]. More recently, the SLOCC classification of the extended GHZ-symmetric states was discussed [19]. Extended GHZ symmetry is the GHZ symmetry without qubit permutation symmetry. It is a larger symmetry group than the usual GHZ symmetry group and is parametrized by four real parameters.

The purpose of this paper is to extend the analysis of Ref. [15] to a four-qubit system. Four-qubit GHZ states[4] (or $m = 4$ cat states, as in Ref. [8]) are defined as

$$|\mathrm{GHZ}_4\rangle_\pm = \frac{1}{\sqrt{2}}(|0000\rangle \pm |1111\rangle). \tag{1.1}$$

Like a three-qubit GHZ symmetry we define a four-qubit GHZ symmetry as a symmetry which $|\mathrm{GHZ}_4\rangle_\pm$ have up to the global phase. Straightforward generalization, which is (i) qubit permutations, (ii) simultaneous flips (i.e., application of $\sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x$), and (iii) qubit rotations about the z axis of the form

$$U(\phi_1,\phi_2,\phi_3) = e^{i\phi_1\sigma_z} \otimes e^{i\phi_2\sigma_z} \otimes e^{i\phi_3\sigma_z} \otimes e^{-i(\phi_1+\phi_2+\phi_3)\sigma_z}, \tag{1.2}$$

is obviously a symmetry of $|\mathrm{GHZ}_4\rangle_\pm$. Thus, we call this symmetry four-qubit GHZ symmetry. As shown later the four-qubit GHZ-symmetric states are represented by three real parameters whereas the three-qubit states contain only two. Thus, it is more difficult to analyze the entanglement classification in the four-qubit GHZ-symmetric case than that in the three–qubit case. Furthermore, if the number of qubit increases, we need real parameters more and more to represent the GHZ-symmetric states. Therefore, classification of the entanglement for the GHZ-symmetric states becomes a formidable task in the higher-qubit system. For this reason it is advisable to restrict the GHZ symmetry to reduce the number of real parameters. This can be achieved by modifying parameter (ii) (simultaneous flips) into (ii) simultaneous and any pair flips without changing parameters (i) and (iii). In a four-qubit system this modification can be stated as an invariance under the application of $\sigma_x \otimes \sigma_x \otimes \mathbb{1} \otimes \mathbb{1}$, $\sigma_x \otimes \mathbb{1} \otimes \sigma_x \otimes \mathbb{1}$, $\sigma_x \otimes \mathbb{1} \otimes \mathbb{1} \otimes \sigma_x$, $\mathbb{1} \otimes \sigma_x \otimes \sigma_x \otimes \mathbb{1}$, $\mathbb{1} \otimes \sigma_x \otimes \mathbb{1} \otimes \sigma_x$, $\mathbb{1} \otimes \mathbb{1} \otimes \sigma_x \otimes \sigma_x$, and $\sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x$. The simplest pure state which has the modified symmetry (ii) is

$$|\psi\rangle_{ABCD} = \frac{1}{2\sqrt{2}}(|0000\rangle + |1100\rangle + |1010\rangle + |1001\rangle + |0110\rangle + |0101\rangle + |0011\rangle + |1111\rangle). \tag{1.3}$$

It is easy to show that $|\psi\rangle_{ABCD}$ is symmetric under the flips of (A,B), (A,C), (A,D), (B,C), (B,D), (C,D), or (A,B,C,D) parties. Obviously, $|\mathrm{GHZ}_4\rangle_\pm$ do not have this modified symmetry. Of course, the states, which have this modified symmetry, are also GHZ symmetric. Therefore, we call this modified symmetry restricted GHZ (RGHZ) symmetry.[5] As shown, the RGHZ-symmetric states are represented by two real parameters like the three-qubit case.

This paper is organized as follows. In Sec. II the general forms of the GHZ- and RGHZ-symmetric states are derived, respectively. It is shown that while the GHZ-symmetric states are represented by three real parameters, the RGHZ-symmetric states are represented by two real parameters. In Sec. III we classify the entanglement of the RGHZ-symmetric states. It is shown that entanglement of the RGHZ-symmetric states is either L_{abc_2} or G_{abcd}. In Sec. IV a brief conclusion is given.

II. GHZ-SYMMETRIC AND RGHZ-SYMMETRIC STATES

In this section we derive the general forms of the GHZ-symmetric and RGHZ-symmetric states and compare them with each other.

A. GHZ-symmetric states

It is not difficult to show that the general form of the GHZ-symmetric states is

$$\rho_4^{\mathrm{GHZ}} = \tilde{x}[|0000\rangle\langle 1111| + |1111\rangle\langle 0000|] + \mathrm{diag}(\tilde{\alpha}_1,\tilde{\alpha}_2,\tilde{\alpha}_2,\tilde{\alpha}_3,\tilde{\alpha}_2,\tilde{\alpha}_3,\tilde{\alpha}_3,\tilde{\alpha}_2,\tilde{\alpha}_2,\tilde{\alpha}_3, \tilde{\alpha}_3,\tilde{\alpha}_2,\tilde{\alpha}_3,\tilde{\alpha}_2,\tilde{\alpha}_2,\tilde{\alpha}_1), \tag{2.1}$$

where $\tilde{x}$, $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, and $\tilde{\alpha}_3$ are real numbers satisfying $\tilde{\alpha}_1 + 4\tilde{\alpha}_2 + 3\tilde{\alpha}_3 = \frac{1}{2}$. Unlike the three-qubit case, ρ_4^{GHZ} is represented by three real parameters.

Now, we define the following two real parameters $\tilde{y}$ and $\tilde{z}$ as

$$\tilde{y} = \mathcal{N}_1[\tilde{\alpha}_1 + (\sqrt{10}+3)\tilde{\alpha}_2], \quad \tilde{z} = \mathcal{N}_2[(\sqrt{10}+3)\tilde{\alpha}_1 - \tilde{\alpha}_2], \tag{2.2}$$

where

$$\mathcal{N}_1 = \sqrt{\frac{2}{3} - \frac{2}{15}\sqrt{10}} \approx 0.495, \quad \mathcal{N}_2 = \sqrt{\frac{14}{3} - \frac{22}{15}\sqrt{10}} \approx 0.169. \tag{2.3}$$

Then, it is straightforward to show that the Hilbert-Schmidt metric of ρ_4^{GHZ} is equal to the Euclidean metric, i.e.,

$$d^2\left[\rho_4^{\mathrm{GHZ}}(\tilde{\alpha}_1,\tilde{\alpha}_2,\tilde{\alpha}_3,\tilde{x}),\rho_4^{\mathrm{GHZ}}(\tilde{\alpha}_1',\tilde{\alpha}_2',\tilde{\alpha}_3',\tilde{x}')\right] = (\tilde{x}-\tilde{x}')^2 + (\tilde{y}-\tilde{y}')^2 + (\tilde{z}-\tilde{z}')^2, \tag{2.4}$$

where $d^2(A,B) = \frac{1}{2}\mathrm{tr}(A-B)^\dagger(A-B)$. The three real parameters $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$ can be represented as

$$\tilde{x} = \frac{1}{2}\left[{}_+\langle \mathrm{GHZ}_4|\rho_4^{\mathrm{GHZ}}|\mathrm{GHZ}_4\rangle_+ - {}_-\langle \mathrm{GHZ}_4|\rho_4^{\mathrm{GHZ}}|\mathrm{GHZ}_4\rangle_-\right],$$

[4] While $|\mathrm{GHZ}_3\rangle_+$ is a unique maximally entangled three-qubit state up to LU, $|\mathrm{GHZ}_4\rangle_+$ is not a unique maximally entangled state. In a four-qubit system there are two additional maximally entangled states $|\Phi_5\rangle = (1/\sqrt{6})(\sqrt{2}|1111\rangle + |1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle)$ and $|\Phi_4\rangle = (1/2)(|1111\rangle + |1100\rangle + |0010\rangle + |0001\rangle$ [20].

[5] The state $|\psi\rangle_{ABCD}$ given in Eq. (1.3) is not RGHZ symmetric because it is not symmetric under the qubit rotation about the z axis although it is symmetric under the modified parameter (ii). In fact, there is no pure RGHZ-symmetric state.

$$\tilde{y} = \frac{\mathcal{N}_1}{2}\big[{}_{+}\langle \mathrm{GHZ}_4|\rho_4^{\mathrm{GHZ}}|\mathrm{GHZ}_4\rangle_{+} + {}_{-}\langle \mathrm{GHZ}_4|\rho_4^{\mathrm{GHZ}}|\mathrm{GHZ}_4\rangle_{-} + 2(\sqrt{10}+3)\langle \Phi|\rho_4^{\mathrm{GHZ}}|\Phi\rangle\big], \tag{2.5}$$

$$\tilde{z} = \frac{\mathcal{N}_2}{2}\big[(\sqrt{10}+3)\,{}_{+}\langle \mathrm{GHZ}_4|\rho_4^{\mathrm{GHZ}}|\mathrm{GHZ}_4\rangle_{+} + (\sqrt{10}+3)\,{}_{-}\langle \mathrm{GHZ}_4|\rho_4^{\mathrm{GHZ}}|\mathrm{GHZ}_4\rangle_{-} - 2\langle \Phi|\rho_4^{\mathrm{GHZ}}|\Phi\rangle\big],$$

where $|\Phi\rangle$ is either $|\Phi^+\rangle = (|0001\rangle + |1110\rangle)/\sqrt{2}$ or $|\Phi^-\rangle = (|0001\rangle - |1110\rangle)/\sqrt{2}$.

In order for ρ_4^{GHZ} to be a physical state the parameters should be restricted to

$$0 \leqslant \tilde{\alpha}_2 \leqslant \tfrac{1}{8}, \quad 0 \leqslant \tilde{\alpha}_3 \leqslant \tfrac{1}{6}, \quad 0 \leqslant \tilde{\alpha}_1 \leqslant \tfrac{1}{2} \tag{2.6}$$

and

$$\tilde{\alpha}_1 \geqslant \pm\tilde{x}. \tag{2.7}$$

These physical conditions imply that any GHZ-symmetric physical state is represented as a point inside a tetrahedron shown in Fig. 1(a). In this figure two black dots represent $|\mathrm{GHZ}_4\rangle_{\pm}$, respectively. It is worthwhile to note that the sign of x does not change the character of entanglement because $\rho_4^{\mathrm{GHZ}}(-\tilde{x},\tilde{y},\tilde{z}) = u\rho_4^{\mathrm{GHZ}}(\tilde{x},\tilde{y},\tilde{z})u^{\dagger}$, where $u = i\sigma_x \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y$.

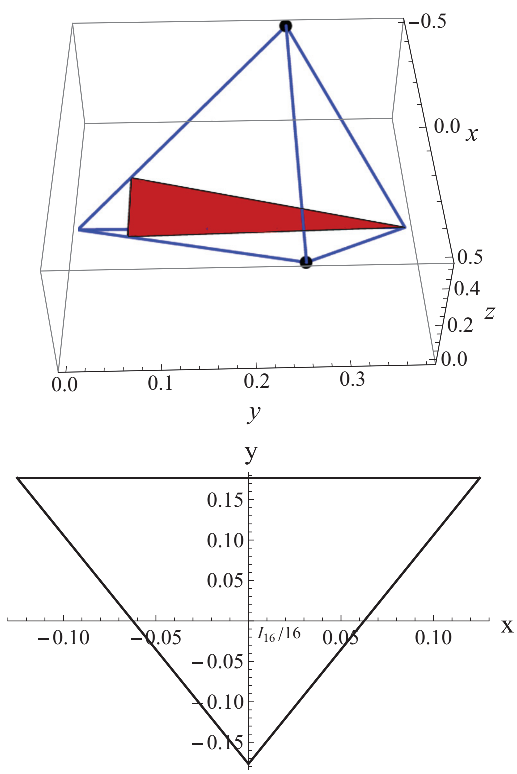

FIG. 1. (Color online) Each point in the tetrahedron corresponds to the GHZ-symmetric state. Two black dots represent the four-qubit GHZ state $|\mathrm{GHZ}_4\rangle_{\pm}$. The surface of the triangle in the tetrahedron is the place where the RGHZ-symmetric states reside. (b) Each point in the triangle corresponds to the RGHZ-symmetric state. This triangle is equivalent to the triangle in (a). Thus, RGHZ-symmetric states make up a very small portion and are of zero measure in the whole set of the GHZ-symmetric states.

B. RGHZ-symmetric states

It is straightforward to show that the general form of RGHZ-symmetric states is

$$\rho_4^{\mathrm{RGHZ}} = x[|0000\rangle\langle 1111| + |1111\rangle\langle 0000|] + \mathrm{diag}(\alpha_1,\alpha_2,\alpha_2,\alpha_1,\alpha_2,\alpha_1,\alpha_1,\alpha_2,\alpha_2,\alpha_1,\alpha_1,\alpha_2,\alpha_1,\alpha_2,\alpha_2,\alpha_1) \tag{2.8}$$

with $\alpha_1 = \frac{1}{16} + \frac{y}{2\sqrt{2}}$ and $\alpha_2 = \frac{1}{16} - \frac{y}{2\sqrt{2}}$. The parameters x and y are chosen such that the Euclidean metric in the (x,y) plane coincides with the Hilbert-Schmidt metric $d^2(A,B) = \frac{1}{2}\mathrm{tr}(A-B)^{\dagger}(A-B)$ again. The parameters can be represented as

$$x = \frac{1}{2}\big[{}_{+}\langle \mathrm{GHZ}_4|\rho_4^{\mathrm{RGHZ}}|\mathrm{GHZ}_4\rangle_{+} - {}_{-}\langle \mathrm{GHZ}_4|\rho_4^{\mathrm{RGHZ}}|\mathrm{GHZ}_4\rangle_{-}\big],$$

$$y = \sqrt{2}\left[{}_{+}\langle \mathrm{GHZ}_4|\rho_4^{\mathrm{RGHZ}}|\mathrm{GHZ}_4\rangle_{+} + {}_{-}\langle \mathrm{GHZ}_4|\rho_4^{\mathrm{RGHZ}}|\mathrm{GHZ}_4\rangle_{-} - \frac{1}{8}\right]. \tag{2.9}$$

It is also worthwhile to note that the sign of x does not change the entanglement because $\rho_4^{\mathrm{RGHZ}}(-x,y) = u\rho_4^{\mathrm{RGHZ}}(x,y)u^{\dagger}$. This is evident from the fact that the RGHZ-symmetric state is also GHZ symmetric.

Since ρ_4^{RGHZ} is a quantum state, it should be a positive operator, which restricts the parameters as follows:

$$y \geqslant \pm 2\sqrt{2}x - \frac{\sqrt{2}}{8}, \quad |x| \leqslant \frac{1}{8}. \tag{2.10}$$

Thus, any RGHZ-symmetric physical state is represented as a point in a triangle depicted in Fig. 1(b).

It is easy to show that ρ_4^{GHZ} is RGHZ symmetric if and only if $\tilde{x} = x$, $\tilde{\alpha}_2 = \alpha_2$, and $\tilde{\alpha_1} = \tilde{\alpha}_3 = \alpha_1$ or equivalently

$$\tilde{y} = \mathcal{N}_1\left[\frac{\sqrt{10}+4}{16} - \frac{\sqrt{10}+2}{2\sqrt{2}}y\right], \quad \tilde{z} = \mathcal{N}_2\left[\frac{\sqrt{10}+2}{16} + \frac{\sqrt{10}+4}{2\sqrt{2}}y\right]. \tag{2.11}$$

Using this relation it is possible to know where the RGHZ-symmetric states reside in the tetrahedron in Fig. 1(a). In this figure the red triangle is equivalent to the one in Fig. 1(b). Thus, the states on this triangle are RGHZ symmetric. From Fig. 1(a) one can realize that the RGHZ-symmetric states make up a very small portion and are of zero measure in the entire set of GHZ-symmetric states.

III. SLOCC CLASSIFICATION OF RGHZ-SYMMETRIC STATES

The SLOCC classification of the four-qubit pure-state system was first discussed in Ref. [21] by making use of the Jordan block structure of some complex symmetric matrix. Subsequently, the same issue was explored in several more papers using different approaches [22]. Unlike, however, two- and three-qubit cases, the results of Refs. [21,22] seem to contradict each other. This means that still our understanding on the four-qubit entanglement is incomplete.

In this paper we adopt the results of Ref. [21], where there are the following nine inequivalent SLOCC classes:

$$\begin{aligned}
G_{abcd} &= \frac{a+d}{2}(|0000\rangle + |1111\rangle) + \frac{a-d}{2}(|0011\rangle \\
&\quad + |1100\rangle) + \frac{b+c}{2}(|0101\rangle + |1010\rangle) \\
&\quad + \frac{b-c}{2}(|0110\rangle + |1001\rangle), \\
L_{abc_2} &= \frac{a+b}{2}(|0000\rangle + |1111\rangle) + \frac{a-b}{2}(|0011\rangle \\
&\quad + |1100\rangle) + c(|0101\rangle + |1010\rangle) + |0110\rangle, \\
L_{a_2b_2} &= a(|0000\rangle + |1111\rangle) + b(|0101\rangle + |1010\rangle) \\
&\quad + |0110\rangle + |0011\rangle, \\
L_{ab_3} &= a(|0000\rangle + |1111\rangle) + \frac{a+b}{2}(|0101\rangle \\
&\quad + |1010\rangle) + \frac{a-b}{2}(|0110\rangle + |1001\rangle) \\
&\quad + \frac{i}{\sqrt{2}}(|0001\rangle + |0010\rangle + |0111\rangle + |1011\rangle), \\
L_{a_4} &= a(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle) \\
&\quad + (i|0001\rangle + |0110\rangle - i|1011\rangle), \\
L_{a_2 0_{3\oplus\bar{1}}} &= a(|0000\rangle + |1111\rangle) + (|0011\rangle + |0101\rangle + |0110\rangle), \\
L_{0_{5\oplus\bar{3}}} &= |0000\rangle + |0101\rangle + |1000\rangle + |1110\rangle, \\
L_{0_{7\oplus\bar{1}}} &= |0000\rangle + |1011\rangle + |1101\rangle + |1110\rangle, \\
L_{0_{3\oplus\bar{1}}0_{3\oplus\bar{1}}} &= |0000\rangle + |0111\rangle,
\end{aligned} \tag{3.1}$$

where a, b, c, and d are complex parameters with non-negative real part. In Eqs. (3.1) G_{abcd} is special in a sense that all its local states are completely mixed. In other words, G_{abcd} is a set of normal states [23].

A. L_{abc_2}

In this section we examine a question where the states of L_{abc_2} reside in the triangle in Fig. 1(b). Before proceeding further, it is important to note that there is a correspondence between four-qubit pure states and RGHZ-symmetric states. Let $|\psi\rangle$ be a four-qubit pure state. Then, the corresponding RGHZ-symmetric state $\rho_4^{\mathrm{RGHZ}}(\psi)$ can be written as

$$\rho_4^{\mathrm{RGHZ}}(\psi) = \int U|\psi\rangle\langle\psi|U^\dagger, \tag{3.2}$$

where the integral is understood to cover the entire RGHZ symmetry group, i.e., unitaries $U(\phi_1,\phi_2,\phi_3)$ in Eq. (1.2) and averaging over the discrete symmetries. For example, if $|\psi\rangle = \sum_{i,j,k,l=0}^{1} \psi_{ijkl}|ijkl\rangle$, $\rho_4^{\mathrm{RGHZ}}(\psi)$ becomes Eq. (2.8) with

$$\begin{aligned}
x &= \frac{1}{4}\mathrm{Re}[\psi_{0000}\psi_{1111}^* + \psi_{0011}\psi_{1100}^* \\
&\quad + \psi_{0101}\psi_{1010}^* + \psi_{0110}\psi_{1001}^*], \\
\alpha_1 &\equiv \frac{1}{16} + \frac{y}{2\sqrt{2}} = \frac{1}{8}[|\psi_{0000}|^2 + |\psi_{1111}|^2 + |\psi_{0011}|^2 + |\psi_{0101}|^2 \\
&\quad + |\psi_{0110}|^2 + |\psi_{1001}|^2 + |\psi_{1010}|^2 + |\psi_{1100}|^2].
\end{aligned} \tag{3.3}$$

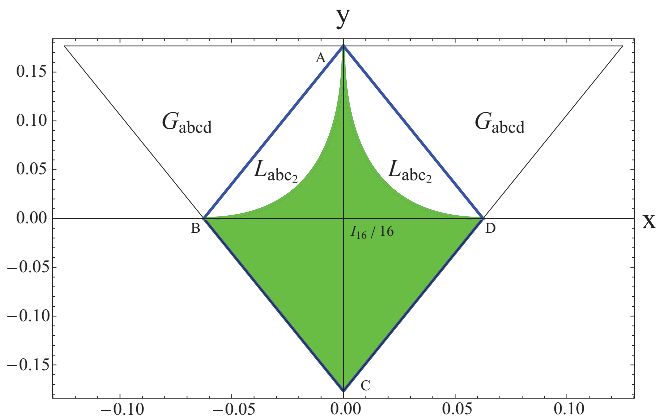

FIG. 2. (Color online) The SLOCC classification of RGHZ-symmetric states ρ_4^{RGHZ}. In the polygon ABCD states of L_{abc_2} reside. Theorem 2 implies that there is no one-qubit tensor product three-qubit entangled states in the RGHZ-symmetric states. This fact implies that the RGHZ symmetry excludes $L_{a_2 0_{3\oplus\bar{1}}}$, $L_{0_{3\oplus\bar{1}}0_{3\oplus\bar{1}}}$, and $L_{a_2b_2}$. Theorem 3 implies that there are states of G_{abcd} outside the polygon ABCD.

Now, we are ready to discuss the main issue of this section.

Theorem 1. The RGHZ-symmetric states of the L_{abc_2} class reside in the polygon ABCD in Fig. 2.

Proof. First we note that when $a = b = c = 0$, L_{abc_2} reduces to the fully separable state $|0110\rangle$. Since LU is a particular case of SLOCC, this fact implies that all fully separable states are in L_{abc_2}. Let $|\psi^{\mathrm{sep}}\rangle = (u_1 \otimes u_2 \otimes u_3 \otimes u_4)|0000\rangle$, where

$$u_j = \begin{pmatrix} A_j & -C_j^* \\ C_j & A_j^* \end{pmatrix} \quad \text{with} \quad |A_j|^2 + |C_j|^2 = 1. \tag{3.4}$$

Then, it is easy to derive the parameters x and y of $\rho_4^{\mathrm{RGHZ}}(\psi^{\mathrm{sep}})$ easily using Eqs. (3.3). Our method for proof is as follows. Applying the Lagrange multiplier method we maximize x with given y. Then, it is possible to derive a boundary $x_{\max} = x_{\max}(y)$ in the (x,y) plane. If a region inside the boundary is convex, this is the region where the L_{abc_2}-class states reside. If it is not convex, we have to choose the convex hull of it for the residential region.

From symmetry it is evident that the maximum of x occurs when $A_1 = A_2 = A_3 = A_4 \equiv A$. Then the constraint of y yields $A^2 = \frac{1}{2}(1 \pm 2^{5/8}y^{1/4})$, which gives

$$x_{\max} = \tfrac{1}{16}(1 - 2^{\frac{5}{4}}y^{\frac{1}{2}})^2. \tag{3.5}$$

Since the sign of $x_{\max}$ does not change the entanglement class, the region represented by green (the shaded region) in Fig. 2 is derived. Since it is not convex, we have to choose a convex hull, which is a polygon ABCD in Fig. 2. This completes the proof.

Although we start with a fully separable state, this does not guarantee that all states in the polygon ABCD are fully separable because L_{abc_2} has four-way entangled states as well as fully separable states. The only fact we can assert is that all L_{abc_2}-class RGHZ-symmetric states reside in the polygon ABCD.

B. $L_{a_2 0_{3\oplus\bar{1}}}, L_{0_{3\oplus\bar{1}} 0_{3\oplus\bar{1}}}, \ldots$

In this section we show that the RGHZ symmetry excludes all SLOCC classes except G_{abcd}.

Theorem 2. There is no one-qubit product GHZ state in the RGHZ-symmetric states.

Proof. Let $|\psi^{\rm GHZ}\rangle = (G_1 \otimes G_2 \otimes G_3 \otimes G_4)|0\rangle \otimes (|000\rangle + |111\rangle)$, where

$$G_j = \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix}. \tag{3.6}$$

Then, it is easy to compute x and y of $\rho_4^{\rm RGHZ}(\psi^{\rm GHZ})$ using Eq. (3.3). Now, we want to maximize x with given y and $\langle\psi^{\rm GHZ}|\psi^{\rm GHZ}\rangle = 1$. From symmetry of the Lagrange multiplier equations it is evident that the maximum of x occurs when $A_2 = A_3 = A_4 = B_2 = B_3 = B_4 \equiv a$ and $C_2 = C_3 = C_4 = D_2 = D_3 = D_4 \equiv c$. Then, we define $x^\Lambda = x + \Lambda_0\Theta_0 + \Lambda_1\Theta_1$, where Λ_0 and Λ_1 are Lagrange multiplier constants, and

$$\begin{aligned} x &= 4A_1C_1a^3c^3, \\ \Theta_0 &= 4\left(A_1^2 + C_1^2\right)(a^2 + c^2)^2 - 1, \\ \Theta_1 &= 4\left[A_1^2a^2(a^4 + 3c^4) + C_1^2c^2(3a^4 + c^4) - 2\alpha_1\right]. \end{aligned} \tag{3.7}$$

Now, we want to maximize x under the constraints $\Theta_0 = \Theta_1 = 0$.

First, we solve the two constraints, whose solutions are

$$A_1^2 = \frac{8\alpha_1(u_1 + u_2) - u_2}{u_1^2 - u_2^2}, \quad C_1^2 = \frac{u_1 - 8\alpha_1(u_1 + u_2)}{u_1^2 - u_2^2}, \tag{3.8}$$

where $u_1 = 4a^2(a^4 + 3c^4)$ and $u_2 = 4c^2(3a^4 + c^4)$. From $\frac{\partial x^\Lambda}{\partial A_1} = \frac{\partial x^\Lambda}{\partial C_1} = 0$ one can express the Lagrange multiplier constants as

$$\begin{aligned} \Lambda_0 &= -\frac{A_1^2u_1 - C_1^2u_2}{A_1C_1}\frac{2a^3c^3}{u_1^2 - u_2^2}, \\ \Lambda_1 &= \frac{A_1^2 - C_1^2}{A_1C_1}\frac{2a^3c^3}{u_1 - u_2}. \end{aligned} \tag{3.9}$$

Combining Eqs. (3.8), (3.9), and $\frac{\partial x^\Lambda}{\partial a} = \frac{\partial x^\Lambda}{\partial c} = 0$, we obtain

$$8\alpha_1(z^2 + 1)^4 = z^8 + 6z^4 + 1, \tag{3.10}$$

where $z = \frac{a}{c}$. Then, the maximum of x with given y becomes

$$x_{\max} = \frac{z^3\sqrt{8\alpha_1(1 - 8\alpha_1)(1 + z^2)^6 - z^2(3 + z^4)(1 + 3z^4)}}{(z^4 - 1)^3}. \tag{3.11}$$

Using Eq. (3.10) and performing long and tedious calculation, one can show that the right-hand side of Eq. (3.11) reduces to $\frac{1}{16}(1 - \sqrt{16\alpha_1 - 1})^2$, which results in the identical equation with Eq. (3.5). This implies that there is no one-qubit product three-qubit GHZ state in the RGHZ-symmetric states. This completes the proof.

From this theorem one can conclude that there is no $L_{0_{3\oplus\bar{1}} 0_{3\oplus\bar{1}}}$ in the RGHZ-symmetric states, because this class involves a one-qubit product GHZ state. Since it is well known that the three-qubit states consist of fully separable (S), biseparable (B), W, and GHZ states, and they satisfy a linear hierarchy $S \subset B \subset W \subset$ GHZ, Theorem 2 also implies that there is no one-qubit product W state in the RGHZ-symmetric states. Thus, RGHZ symmetry excludes $L_{a_2 0_{3\oplus\bar{1}}}$ too because this class contains the one-qubit product W state when $a = 0$. This theorem also implies that there is no one-qubit product B state, which excludes $L_{a_2 b_2}$. Similarly, one can exclude all classes except G_{abcd} class.[6]

C. G_{abcd}

Now, we want to discuss the entanglement classes of the remaining RGHZ-symmetric states. In order to conjecture the classes quickly, let us consider the following double biseparable state:

$$|\psi^{BB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{3.12}$$

Such a state belongs to G_{abcd} with ($a = 1, b = c = d = 0$) or $a = b = c = d$. Then, Eq. (3.3) shows that the parameters of $\rho_4^{\rm GHZL}(\psi^{BB})$ are $x = 1/8$ and $y = \sqrt{2}/8$, which correspond to the upper right corner of the triangle in Fig. 2. Since mixing can result only in the same or a lower entanglement class, the entanglement class of this corner state should be G_{abcd} or its subclasses. However, the subclass of this state should be a class where fully separable states belong, and those states are confined in $ABCD$. Therefore, the corner should be G_{abcd}. This fact strongly suggests that all remaining states in Fig. 2 are G_{abcd}. The following theorem shows that our conjecture is correct.

Theorem 3. All remaining RGHZ-symmetric states in Fig. 2 are G_{abcd} class.

Proof. Let $|\psi^{BB}\rangle = (G_1 \otimes G_2 \otimes G_3 \otimes G_4)(|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle)$, where G_j is given in Eq. (3.6). Then, it is easy to compute the parameters x and y of $\rho_4^{\rm GHZL}(\psi^{BB})$ using Eq. (3.3). Similar to the previous theorems we want to maximize x with given y. From a symmetry of Lagrange multiplier equations it is evident that the maximum of x occurs when

$$\begin{aligned} A_1 &= A_2 \equiv a_1, \quad A_3 = A_4 \equiv a_3, \\ B_1 &= B_2 \equiv b_1, \quad B_3 = B_4 \equiv b_3, \\ C_1 &= C_2 \equiv c_1, \quad C_3 = C_4 \equiv c_3, \\ D_1 &= D_2 \equiv d_1, \quad D_3 = D_4 \equiv d_3. \end{aligned} \tag{3.13}$$

For later convenience we define $\mu_1 = a_1^2 + b_1^2$, $\mu_2 = a_3^2 + b_3^2$, $\mu_3 = c_1^2 + d_1^2$, $\mu_4 = c_3^2 + d_3^2$, $\nu_1 = a_1c_1 + b_1d_1$, and $\nu_2 = a_3c_3 + b_3d_3$.

[6]For other classes it is easier to adopt the following numerical calculation than applying the Lagrange multiplier method. First, we select a representative state $|\psi\rangle$ for each SLOCC class. Next, we generate 16 random numbers and identify them with A_j, B_j, C_j, D_j $(j = 1, \ldots, 4)$. Then, using a mapping (3.3) one can compute x and y for pure state $G_1 \otimes G_2 \otimes G_3 \otimes G_4|\psi\rangle$. Repeating this procedure over and over and collecting all (x, y) data, one can deduce numerically the residential region of this class. The numerical calculation shows that the residence of all SLOCC classes except G_{abcd} is confined in the polygon ABCD of Fig. 2.

In order to apply the Lagrange multiplier method we define $x^\Lambda = x + \Lambda_0\Theta_0 + \Lambda_1\Theta_1$, where

$$\begin{aligned} x &= \tfrac{1}{2}\left(\mu_1\mu_2\mu_3\mu_4 + \nu_1^2\nu_2^2\right), \\ \Theta_0 &= \left(\mu_1^2 + 2\nu_1^2 + \mu_3^2\right)\left(\mu_2^2 + 2\nu_2^2 + \mu_4^2\right) - 1, \\ \Theta_1 &= \left(\mu_1^2 + \mu_3^2\right)\left(\mu_2^2 + \mu_4^2\right) + 4\nu_1^2\nu_2^2 - 8\alpha_1. \end{aligned} \tag{3.14}$$

The constraints $\Theta_0 = 0$ and $\Theta_1 = 0$ come from $\langle\psi^{BB}|\psi^{BB}\rangle = 1$ and Eq. (3.3), respectively.

Now, we have eight equations $\frac{\partial x^\Lambda}{\partial \mu_i} = 0$ $(i = 1,2,3,4)$, $\frac{\partial x^\Lambda}{\partial \nu_i} = 0$ $(i = 1,2)$, and $\Theta_0 = \Theta_1 = 0$. Analyzing those equations, one can show that the maximum of x occurs when $\mu_1 = \mu_3$ and $\mu_2 = \mu_4$. Then, the constraint $\Theta_1 = 0$ implies

$$x_{\max} = \frac{1}{16} + \frac{y}{2\sqrt{2}}, \tag{3.15}$$

which corresponds to the right-hand side of the triangle in Fig. 2. This fact implies that all the RGHZ-symmetric states are G_{abcd} or its subclass. Since L_{abc_2} are confined in the polygon ABCD and the remaining classes except G_{abcd} are already excluded, the states outside the polygon ABCD should be G_{abcd} class, which completes the proof.

Although we start with a double biseparable state, this fact does not imply that all states outside the polygon are double biseparable because G_{abcd} contains four-way entangled states as well as double biseparable states. The only fact we can say is that all states outside the polygon ABCD are G_{abcd} class.

IV. CONCLUSION

In this paper the GHZ and RGHZ symmetries in a four-qubit system are examined. It is shown that the whole group of RGHZ-symmetric states involves only two SLOCC classes, L_{abc_2} and G_{abcd}. Following Ref. [17] we can use our result to construct the optimal witness $\mathcal{W}_{G_{abcd}\setminus L_{abc_2}}$, which can detect the G_{abcd} class optimally from a set of L_{abc_2} plus G_{abcd} states.

As remarked earlier if we choose GHZ symmetry, the symmetric states are represented by three real parameters as Eq. (2.1) shows. Probably, these symmetric states involve more kinds of the four-qubit SLOCC classes. The SLOCC classification of Eq. (2.1) will be explored in the future.

Another interesting extension of this paper is to generalize our analysis to any $2n$-qubit system. Then, our modification of symmetry should be changed into "any one-pair, two-pair, …, and n-pair flips." This would drastically reduce the number of free parameters in the set of symmetric states. This strongly restricted symmetry may shed light on the SLOCC classification of the multipartite states.

ACKNOWLEDGMENT

This work was supported by the Kyungnam University Foundation Grant, 2013.

[1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009) and references therein.
[2] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[3] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).
[4] V. Scarani, S. Lblisdir, N. Gisin, and A. Acin, Rev. Mod. Phys. **77**, 1225 (2005) and references therein.
[5] A. K. Ekert, Phys. Rev. Lett. **67**, 661 (1991).
[6] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, U.K., 2000).
[7] G. Vidal, Phys. Rev. Lett. **91**, 147902 (2003).
[8] C. H. Bennett, S. Popescu, D. Rohrlich, J. A. Smolin, and A. V. Thapliyal, Phys. Rev. A **63**, 012307 (2000).
[9] G. Vidal, J. Mod. Opt. **47**, 355 (2000).
[10] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000).
[11] V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A **61**, 052306 (2000).
[12] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[13] A. Acín, D. Bruß, M. Lewenstein, and A. Sanpera, Phys. Rev. Lett. **87**, 040401 (2001).
[14] R. Lohmayer, A. Osterloh, J. Siewert, and A. Uhlmann, Phys. Rev. Lett. **97**, 260502 (2006); C. Eltschka, A. Osterloh, J. Siewert, and A. Uhlmann, New J. Phys. **10**, 043014 (2008); E. Jung, M.-R. Hwang, D. K. Park, and J.-W. Son, Phys. Rev. A **79**, 024306 (2009); E. Jung, D. K. Park, and J.-W. Son, *ibid.* **80**, 010301(R) (2009); E. Jung, M. R. Hwang, D. K. Park, and S. Tamaryan, Quantum Inf. Comput. **10**, 0377 (2010).
[15] C. Eltschka and J. Siewert, Phys. Rev. Lett. **108**, 020502 (2012).
[16] J. Siewert and C. Eltschka, Phys. Rev. Lett. **108**, 230502 (2012).
[17] C. Eltschka and J. Siewert, Quantum Inf. Comput. **13**, 0210 (2013).
[18] C. Eltschka and J. Siewert, Phys. Rev. A **89**, 022312 (2014).
[19] Eylee Jung and DaeKil Park, Quantum Inf. Comput. **14**, 0937 (2014).
[20] A. Osterloh and J. Siewert, Quantum Inf. Comput. **04**, 531 (2006).
[21] F. Verstraete, J. Dehaene, B. De Moor, and H. Verschelde, Phys. Rev. A **65**, 052112 (2002).
[22] L. Lamata, J. León, D. Salgado, and E. Solano, Phys. Rev. A **75**, 022318 (2007); Y. Cao and A. M. Wang, Eur. Phys. J. D **44**, 159 (2007); O. Chterental and D. Z. Djoković, in *Linear Algebra Research Advances*, edited by G. D. Ling (Nova Science Publishers, Hauppauge, NY, 2007), Chap. 4, pp. 133-167; D. Li, X. Li, H. Huang, and X. Li, Quantum Inf. Comput. **9**, 0778 (2009); S. J. Akhtarshenas and M. G. Ghahi, arXiv:1003.2762; L. Borsten, D. Dahanayake, M. J. Duff, A. Marrani, and W. Rubens, Phys. Rev. Lett. **105**, 100507 (2010).
[23] F. Verstraete, J. Dehaene, and B. De Moor, Phys. Rev. A **68**, 012103 (2003).